Enumeration of chord diagrams

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Abstract

We determine the number of nonequivalent chord diagrams of order n under the action of two groups, C_{2n}, a cyclic group of order $2n$, and D_{2n}, a dihedral group of order $4n$. Asymptotic formulas are also established.

§1

Given $2n$ different points on a circle ask a question: in how many different ways may the points be joined by chords. The answer depends of course on our understanding of the word "different".

The configuration (actually a graph) consisting of the circle and n chords joining $2n$ different point is called a chord diagram of order n or, shortly, n-diagram. In the present paper, we let a group G act on the circle and consider two n-diagrams as indistinguishable or equivalent if the one is transformed into the other by a suitable element of the group.

With the identity group acting on the circle, all n-diagrams are distinct and there are altogether $\frac{(2n)!}{2^n n!} = (2n-1)!!$ diagrams with n chords. This case is well-studied: Errera [1] (see also A. M. Jaglom and I. M. Jaglom [2]) determined the number of n-diagrams with the additional requirement that no chords intersect inside the circle which equals $\frac{(2n)!}{n!(n+1)!}$. Touchard [3] and later Riordan [4] extended that result to enumeration of n-diagrams by the number of crossings of the chords, which is given by a generating function $T_n(x)$ satisfying the following relation

$$(1-x)^n T_n(x) = \sum_{j=0}^{n} (-1)^j t_{nj} x^J$$

with

$$J = \binom{j+1}{2}, \quad t_{nj} = \frac{2j+1}{2n+1}\binom{2n+1}{n-j}.$$

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Generally chord diagrams are not strict graphs as they may have double edges. Chord diagrams which are strict graphs were considered by Hazewinkel and Kalashnikov [5]. Let b_{2n} be the number of strict n-diagrams under the identity group. They proved that $a_{2n} = \sum_{i=1}^{n} b_{2i}$ satisfies the recurrence $a_{2n} = (2n-1)\, a_{2n-2} + a_{2n-4}$.

Chord diagrams considered to within an equivalence induced by a cyclic group action appear in different contexts. In the theory of knots they are used to describe Vassiliev knot invariants in a purely combinatorial way by considering the algebra of functions defined on the set of chord diagrams satisfying certain linear equations [6], [7]. Chord diagrams also appear in the classification of vector fields and smooth functions on surfaces up to a homeomorphism where stable separatrices of critical points together with a component of the surface boundary form a chord diagram embedded into the surface [8].

§2

Definition 1. *A chord diagram of order n is a 3-regular graph with the vertex set $[2n] = \{1, 2, \ldots, 2n\}$ containing the $2n$-circuit $\Delta_{2n} = (1\,2\,\ldots\,2n)$ as a subgraph. The circuit is called a circle, the edges not belonging to the circuit are called chords.*

Definition 2. *Let a group G act on the circuit Δ_{2n}. Two n-diagrams Γ_1 and Γ_2 are said to be equivalent if there is a $g \in G$ which takes the chords of Γ_1 into the chords of Γ_2.*

Our first result is the following

Theorem 1. *The number of nonequivalent n-diagrams under the action of a group G equals*

$$\frac{1}{|S_n \wr S_2| \cdot |G|} \sum_{\pi \in S_n \wr S_2} \sum_{\eta \in G} \prod_i i^{\pi_i}\, \eta_i\,(\eta_i - 1)\, \ldots\, (\eta_i - \pi_i + 1). \tag{1}$$

Here $S_n \wr S_2$ is the wreath product of two symmetric groups S_n and S_2, $\pi \in S_n \wr S_2$ has cycle type $1^{\pi_1} 2^{\pi_2} \ldots (2n)^{\pi_{2n}}$, $\eta \in G$ cycle type $1^{\eta_1} 2^{\eta_2} \ldots (2n)^{\eta_{2n}}$ and the product is taken over all $i \in [2n]$ such that $\pi_i > 0$, the product being equal to zero if $\pi_i > \eta_i$ for some i.

Proof. For each chord diagram, a subgraph consisting of its chords is a 1-factor. The chords of all n-diagrams constitute the complete graph K_{2n}, the chords of each single n-diagram being again a 1-factor of K_{2n}. The action of

G on Δ_{2n} induces an action on the set $\mathcal{F}_1$ of all 1-factors of K_{2n}. The orbits of $\mathcal{F}_1$ under that action are in a one-one correspondence with the nonequivalent n-diagrams.

A 1-factor of K_{2n} can be represented by a $n \times 2$ matrix whose entries belong to $[2n]$ and are all distinct: each row corresponds to an edge, two row entries being the end points of the edge. This correspondence is not unique. It is defined to whithin an equivalence induced on the set of such matrices by independently permuting entries in each row and permuting the rows bodily. This amounts to the action of the wreath product $S_n \wr S_2$. The action of G on the set of 1-factors is equivalent to the action of G on the set of $2 \times n$-matrices.

We thus arrive at the following setting: given two sets $[n] \times [2]$ and $[2n]$, consider the set of bijective mappings $[n] \times [2] \to [2n]$. The wreath product $S_n \wr S_2$ acts on the set $[n] \times [2]$ by the rule $(\tau, \bar{\sigma}) \cdot (i, j) = (\tau(i), \sigma_i(j))$ where $\tau \in S_n$, $\bar{\sigma} = (\sigma_1, \dots \sigma_n) \in S_2^n$, and the group G acts on the set $[2n]$. Two mappings $f_1, f_2 : [n] \times [2] \to [2n]$ are equivalent if there exist a $\pi \in S_n \wr S_2$ and an $\eta \in G$ such that

$$f_1(\pi(i, j)) = \eta f_2((i, j))$$

for all $(i, j) \in [n] \times [2]$. The equivalence classes of mappings are in a one-one correspondence with the orbits of $\mathcal{F}_1$. In this setting, an argument of de Bruijn [9] applies which he used to prove a theorem on the number of classes of bijective mappings. The proofe is complete. $\square$

§3

We now specialize G to a cyclic group C_{2n} of order $2n$ and obtain a much simpler expression for the number of nonequivalent n-diagrams.

Theorem 2. *The number c_n of nonequivalent n-diagrams under the action of a cyclic group C_{2n} equals*

$$c_n = \frac{1}{2n} \sum_{i|2n} \varphi(i) \nu_n(i), \tag{2}$$

where $\varphi(i)$ is the Euler function and

$$\nu_n(i) = \begin{cases} i^{n/i} \, (2n/i - 1)!! & i \text{ odd}, \\ \displaystyle\sum_{k=0}^{\lfloor \frac{n}{i} \rfloor} \binom{2n/i}{2k} i^k \, (2k-1)!! & i \text{ even} \end{cases} \tag{3}$$

for $i \mid 2n$.

Proof. Each permutation $\eta \in C_{2n}$ has cycle type $i^{2n/i}$, $i \mid 2n$. If $\pi \in S_n \wr S_2$ has a cycle of length $\neq i$ the product in (1) equals zero, otherwise it reduces to a single term $i^{\pi_i}\eta_i(\eta_i - 1)\dots(\eta_i - \pi_i + 1) = i^{2n/i}(2n/i)!$ since $\pi_i = \eta_i = 2n/i$. The double sum in (1) can then be replaced with a single sum over all $i \mid 2n$. The group C_{2n} contains $\varphi(i)$ permutations of cycle type $i^{2n/i}$. Denoting by $\psi_n(i)$ the number of permutations of the same cycle type $i^{2n/i}$ in $S_n \wr S_2$ we rewrite (1) as follows

$$c_n = \frac{1}{2^n\, n!\, 2n} \sum_{i|2n} i^{2n/i}\,(2n/i)!\,\varphi(i)\,\psi_n(i). \tag{4}$$

To determine $\psi_n(i)$ we first establish a relationship between the cycles of $\tau \in S_n$ and the cycles of $(\tau, \bar{\sigma}) \in S_n \wr S_2$.

If $K_\pi \subset [n] \times [2]$ is a cycle of $\pi = (\tau, \bar{\sigma})$ then its projection onto $[n]$ is a cycle of τ.

If $K_\tau \subset [n]$ is a cycle of $\tau \in S_n$ then, for any $\bar{\sigma} \in S_2^n$, the length of a cycle of $(\tau, \bar{\sigma}) \in S_n \wr S_2$ induced by K_τ depends only on $\sigma_k \in S_2$, $k \in K_\tau$. Let $L = \{k \in K_\tau : \sigma_k \neq e\}$ where e is the identity permutation. If $|L|$ is odd then K_τ induces one cycle of length $2\,|K_\tau|$. If $|L|$ is even then K_τ induces two cycles both of length $|K_\tau|$.

Let $i \mid 2n$ be odd. For $\pi = (\tau, \bar{\sigma})$ to have cycle type $i^{2n/i}$ the permutation τ must have cycle type $i^{n/i}$. The number of such $\tau \in S_n$ equals

$$\frac{n!}{i^{n/i}\,(n/i)!}.$$

Now we fix τ and count the number of $\bar{\sigma} \in S_2^n$ such that $(\tau, \bar{\sigma})$ has cycle type $i^{2n/i}$. Each cycle K_τ of τ induces two cycles of $(\tau, \bar{\sigma})$ of the same length i. Hence $|L|$ must be even for each K_τ. There are

$$\sum_{m \leq i,\, m \text{ even}} \binom{i}{m} = 2^{i-1}$$

choices for $L \subset K_\tau$. Clearly, L uniquely determines σ_k for all $k \in K_\tau$. For different cycles of τ the choices are independent, so we have $\left(2^{i-1}\right)^{n/i} = 2^{n-n/i}$ different $\bar{\sigma} \in S_2^n$. Multiplying the expressions for τ and $\bar{\sigma}$ we get

$$\psi_n(i) = \frac{2^n n!}{2^{n/i}\, i^{n/i}\,(n/i)!} \tag{5}$$

for i odd.

Let now $i \mid 2n$ be even. For $\pi = (\tau, \bar\sigma)$ to have cycle type $i^{2n/i}$ the permutation τ must have cycle type $(i/2)^l i^k$ with $l \geq 0, k \geq 0, l \cdot i/2 + k \cdot i = n$. The number of such $\tau \in S_n$ equals

$$\frac{n!}{(i/2)^l\, l!\, i^k\, k!}.$$

We fix τ and count the number of corresponding $\bar\sigma$. Each $i/2$-cycle of τ induces one cycle of $(\tau, \bar\sigma)$ of the length i. Hence $|L|$ must be odd for each $i/2$-cycle of τ. There are

$$\sum_{m \leq i/2,\, m \text{ odd}} \binom{i/2}{m} = 2^{i/2-1}$$

choices for L. Each i-cycle of τ induces two cycles of $(\tau, \bar\sigma)$, both having length i. Hence $|L|$ must be even for each i-cycle of τ, and we again have 2^{i-1} choices for L. It follows, there are $\left(2^{i/2-1}\right)^l \cdot \left(2^{i-1}\right)^k = 2^{n-l-k}$ different $\bar\sigma \in S_2^n$ such that $(\tau, \bar\sigma)$ has cycle type $i^{2n/i}$. Multiplying the expressions for τ and $\bar\sigma$ and summing up over all admissible l, k we obtain

$$\psi_n(i) = \sum_{\substack{l \geq 0, k \geq 0 \\ l \cdot i/2 + k \cdot i = n}} \frac{2^n\, n!}{2^l\, (i/2)^l\, l!\, 2^k\, i^k\, k!} = \frac{2^n\, n!}{i^{2n/i}} \sum_{k=0}^{\lfloor \frac{n}{i} \rfloor} \frac{i^k}{(2n/i - 2k)!\, 2^k\, k!} \tag{6}$$

for i even.

It remains to substitute (5) and (6) into (4). Setting

$$\nu_n(i) = \frac{i^{2n/i}\, (2n/i)!}{2^n\, n!}\, \psi_n(i)$$

we have

$$\nu_n(i) = \frac{i^{n/i}\, (2n/i)!}{2^{n/i}\, (n/i)!} = i^{n/i}\, (2n/i - 1)!!$$

for i odd and

$$\nu_n(i) = \sum_{k=0}^{\lfloor \frac{n}{i} \rfloor} \frac{i^k\, (2n/i)!}{(2n/i - 2k)!\, 2^k\, k!} = \sum_{k=0}^{\lfloor \frac{n}{i} \rfloor} i^k \binom{2n/i}{2k} (2k-1)!!$$

for i even which completes the proof. $\square$

As the terms in (2) are all positive it is clear that $\underline{c}_n = (2n)^{-1}\, (2n-1)!!$ is a lower bound for c_n for $n \geq 1$. It is an easy matter to show that $\underline{c}_n$ is actually an asymptotic estimate for c_n as $n \to \infty$.

Corollary 1.

$$c_n \sim \underline{c}_n \quad \textit{as } n \to \infty \tag{7}$$

Proof. Dividing out the first term in (2) gives

$$2nc_n = (2n-1)!! + \sum_{i|2n,\ i>1} \varphi(i)\nu_n(i). \tag{8}$$

We begin with establishing an upper bound for $\nu_n(i)$ using Stirling's formula in the following form

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}}$$

where $\theta = \theta(n)$ satisfies $0 < \theta < 1$ (see [10]). We have

$$(2k-1)!! = \sqrt{2}\left(\frac{2k}{e}\right)^k e^{\left(\frac{1}{2}\theta_1 - \theta_2\right)\frac{1}{12k}} < \sqrt{2e}\left(\frac{2k}{e}\right)^k$$

for $k \geq 1$, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$. Then we find

$$\nu_n(i) = (2n/i - 1)!! < \sqrt{2e}\left(\frac{2n}{ei}\right)^{n/i} \tag{9}$$

for $i \mid 2n$ odd. Using the following estimate

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

(see [10]) for the binomal coefficients we obtain

$$\binom{2n/i}{2k} i^k\,(2k-1)!! < \sqrt{2e}\left(\frac{2n^2e}{ik}\right)^k < \sqrt{2e}\big(2en\big)^{n/i}$$

for $k \in \{1, \ldots, \lfloor n/i \rfloor\}$ whence

$$\nu_n(i) = 1 + \sum_{k=1}^{\lfloor n/i \rfloor} \binom{2n/i}{2k} i^k\,(2k-1)!! < \sqrt{2e}\,n\big(2en\big)^{n/i} \tag{10}$$

for $i \mid 2n$ even. Comparing upper bounds (9) and (10) we conclude that

$$\nu_n(i) < \overline{\nu}_n = \sqrt{2e}\,n\big(2en\big)^{n/2}$$

for $i > 1$, $i \mid 2n$. Going back to (8) we see that

$$\sum_{i|2n,\ i>1} \varphi(i)\nu_n(i) < 2n\,\overline{\nu}_n = o\big((2n-1)!!\big)$$

as $n \to \infty$ and the result follows. □

Corollary 1 shows that asymptotically each equivalence class contains $2n$ diagrams, which is equivalent to saying that the fraction of 1-factors of K_{2n} with a nontrivial stabilizer in C_{2n} tends to zero as $n \to \infty$.

§4

The analysis done in the proof of Theorem 2 allows us to handle the case of a dihedral group D_{2n}.

Theorem 3. *The number d_n of nonequivalent n-diagrams under the action of a dihedral group D_{2n} equals*

$$d_n = \frac{1}{2}\big(c_n + \frac{1}{2}\big(\kappa_{n-1} + \kappa_n\big)\big) \tag{11}$$

where

$$\kappa_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!\,(n-2k)!}.$$

Proof. As $C_{2n} < D_{2n}$ it follows from (1) that

$$d_n = \frac{1}{2}\left(c_n + \frac{1}{2^n\, n!\, 2n}\,\gamma_n\right) \tag{12}$$

where γ_n represents the contribution of those permutations of D_{2n} which are not in C_{2n}. Such permutations have cycle type either 2^n or $1^2\,2^{n-1}$ and there are n permutations of each type in D_{2n}. The product in (1) equals $2^n\,n!$ for permutations of cycle type 2^n and $2^n\,(n-1)!$ for permutations of cycle type $1^2\,2^{n-1}$. So we can write

$$\gamma_n = 2^n\,n!\,n\,\psi_n(2) + 2^n\,(n-1)!\,n\,\Psi_n \tag{13}$$

where Ψ_n is the number of $\pi \in S_n \wr S_2$ of cycle type $1^2\,2^{(n-1)}$ and $\psi_n(2)$ is the number of π of cycle type 2^n. Applying the analysis in the proof of Theorem 2 we see that for each $\tau \in S_n$ of cycle type $1^l\,2^k$, $l + 2k = n$, $l \geq 1$, $k \geq 0$ there are $l \cdot 2^k$ permutations $\overline{\sigma} \in S_2^n$ such that $\pi = (\tau, \overline{\sigma}) \in S_n \wr S_2^n$ has cycle type $1^2\,2^{(n-1)}$. Multiplying and summing up over all admissible l, k and simplifying we get

$$\Psi_n = \sum_{\substack{l\geq 1, k\geq 0\\ l+2k=n}} \frac{l\,n!}{l!\,k!} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{(n-1-2k)\,k!}.$$

Substituting the expressions for $\psi_n(2)$ and Ψ_n into (13) we obtain

$$\gamma_n = 2^n\,n!\,n\left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!\,k!} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1)!}{(n-1-2k)!\,k!}\right). \tag{14}$$

Denoting

$$\kappa_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!\,(n-2k)!}$$

and substituting (14) into (12) we get the statement of the theorem. $\square$

As in the case of the cyclic group, $\underline{d}_n = (4n)^{-1}\,(2n-1)!!$ is a lower bound for d_n for $n \geq 1$ and in fact an asymptotic estimate.

Corollary 2.

$$d_n \sim \underline{d}_n \quad \text{as } n \to \infty \tag{15}$$

Proof. From (11) we get

$$4nd_n = 2nc_n + n(\kappa_{n-1} + \kappa_n).$$

Clearly

$$n(\kappa_{n-1} + \kappa_n) < 2n\kappa_n < 2n\,n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} < 2n^2\,n!$$

for $n \geq 1$. Stirling's formula shows that $2n^2\,n! = o\,((2n-1)!!)$ and hence $n(\kappa_{n-1} + \kappa_n) = o\,((2n-1)!!)$. But $2nc_n \sim (2n-1)!!$ which completes the proof. $\square$

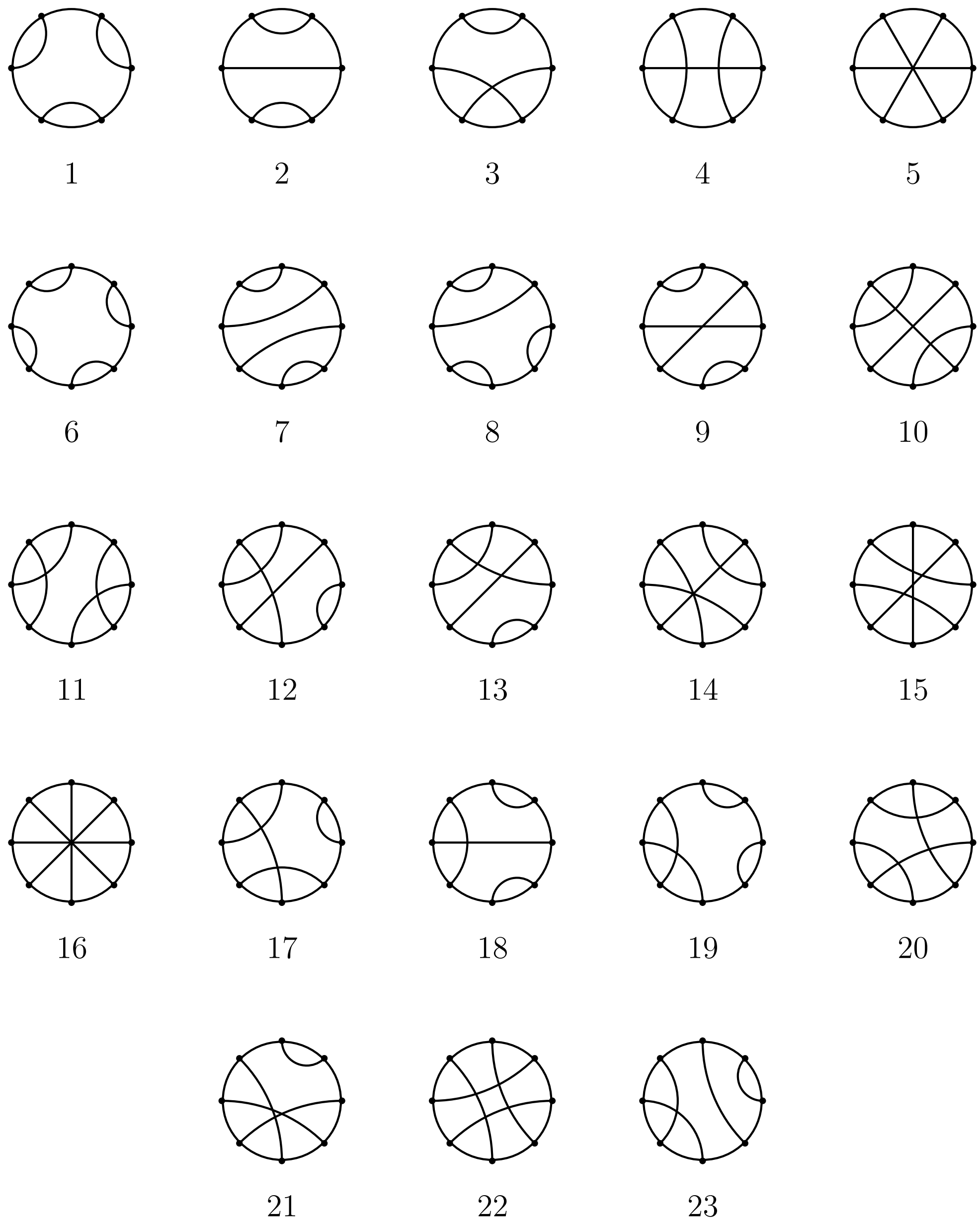
Corollary 2 shows that asymptotically each equivalence class contains $4n$ diagrams.

§5

The following table gives an idea of the growth rate of c_n and d_n along with the integral parts of the corresponding asymptotic estimates.

n	c_n	$\lfloor \underline{c}_n \rfloor$	d_n	$\lfloor \underline{d}_n \rfloor$
3	5	2	5	1
4	18	13	17	6
5	105	94	79	47
6	902	866	554	433
7	9749	9652	5283	4826
8	127072	126689	65346	63344
9	1915951	1914412	966156	957206
10	32743182	32736453	16411700	16368226
11	625002933	624968662	312702217	312484331

Below are shown all nonequivalent (under the cyclic group) 3- and 4-diagrams. Except for the two diagrams 12 and 13 all of them are also nonequivalent under the dihedral group.



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